EM 214

2.4. Exact Differential Equations

An exact DE is one of the form

$$A(x,y)\,dx + B(x,y)\,dy = 0$$

where $A_y = B_x$. The solution is

$$f(x,y) = C$$

where

$$f_x = A \,, \quad f_y = B.$$

NB: such an f exists only when $A_y = B_x$.

<u>Example</u> For each of the following DE's, state whether it is exact or not. If it is exact, solve it.

a) [2.4.7] $(x^2-y^2)dx + (x^2-2xy)dy = 0$

(Ans: not exact)

b) [2.4.21] $(x+y)^2 dx + (2xy + x^2 - 1)dy = 0, \ y(1) = 1$

(Ans: $\frac{1}{3}x^3 + x^2y + xy^2 - y = 4/3$)

c) [2.4]
Similar to 24

$$\left(\frac{3y^2 - t^2}{y^5}\right)\frac{dy}{dt} + \frac{t}{2y^4} = 0, \ y(2) = 1$$

(Ans: $\dfrac{t^2}{4y^4} - \dfrac{3}{2y^2} = -\frac{1}{2}$)

d) $(2xy - \sin x)dx + (x^2 - \cos y)dy = 0$

(Ans: $x^2y + \cos x - \sin y + C = 0$)

Integrating factors

If

$$\omega = A\,dx + B\,dy$$

is not exact, it is sometimes possible to find an <u>integrating factor</u> $g(x)$ or $g(y)$ such that $g\omega$ <u>is</u> exact, ie.

$$gA\,dx + gB\,dy \qquad \text{is exact.}$$

How to find g?

<u>Example</u> a) Find an integrating factor g which will make

$$xy\,dx + (2x^2 + 3y^2 - 20)\,dy$$

exact.

(Answer: try $g(x)$: no. try $g(y)$: yes. $g(y) = y^3$)

b) Solve the differential equation

$$xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0, \quad \text{At } x=0,\ y=1:$$

(Ans: $f = \frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = \frac{9}{2}$)

Example Solve:

$$y(x+y+1)\,dx + (x+2y)\,dy = 0$$

(soln: $e^{x}(xy + y^2) = C$, I.F. is $g(x)$).